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ANALYSIS OF TRANSONIC FLOWS PAST SOLIDS OF REVOLUTION

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We present a numerical iterative scheme for solving gasdynamic problems by the ascertainment method suitable for computing transonic flows past solids of revolution. A short description of the numerical procedure is followed by the results of computing flows past a sphere, an ellipsoid, a combination of a sphere and a cylinder of varying aspect ratio and a combination of a sphere and a cone, for various supercritical values of the Mach number. Mach number level curves constructed illustrate the flow in the local supersonic zones, their configuration and change, and the position of the shock waves.

Numerical methods for analyzing transonic flows in which closed supersonic zones appear are only beginning to be developed. Chushkin [1] used the method of integral correlations to analyze the flow past an ellipsoid of revolution for one particular case, namely when the Mach number of the incident flow is equal to unity and the influence domain is bounded downstream by the limit characteristics. Below we study the possibility of computing transonic flows past solids of revolution using the ascertainment method. The scheme of implicit differences utilized here is described in detail in [2], where it is used to solve a simple problem of the Laval nozzle.

1. To apply the ascertainment method we take the equations of unsteady motion of a perfect gas in cylindrical coordinates xy. They can be written in abbreviated form as

$$\frac{\partial Z}{\partial t} + A\frac{\partial Z}{\partial x} + B\frac{\partial Z}{\partial y} + F = 0 \tag{1.1}$$

Here Z and F are vectors (columns) with the following components

$$Z = \{p, u, v, p\rho^{-\varkappa}\}, \qquad F = \{\varkappa\rho a^2 v y^{-1}, 0, 0, 0\}$$

and the matrices A and B are given by

$$A = \left\| \begin{matrix} u & \varkappa\rho a^2 & 0 & 0 \\ \varkappa^{-1}\rho^{-1} & u & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{matrix} \right\|, \qquad B = \left\| \begin{matrix} v & 0 & \varkappa\rho a^2 & 0 \\ 0 & v & 0 & 0 \\ \varkappa^{-1}\rho^{-1} & 0 & v & 0 \\ 0 & 0 & 0 & v \end{matrix} \right\|$$

Pressure p, density ρ and the speed of sound a are relative to their values in an unperturbed flow. Cylindrical components u and v of the velocity vector are referred to the speed of sound at infinity. The coordinates x and y and time t are also assumed dimensionless and $\varkappa$ denotes the Poisson adiabatic exponent.

We formulate the mixed boundary value problem for the initial hyperbolic system (1.1) as follows. We prescribe a no-leakage condition at the axis of symmetry and at the streamlined solid of revolution $y = Y(x)$. At infinity the velocity and pressure assume their values in unperturbed flow. We can use any, sufficiently reasonable parametric field, as the initial conditions.

We shall carry out the numerical solution of the problem in new variables $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ which map the meridional section of the flow onto the interior of a unit square. Since we construct a net of constant size, h_1 in ξ and h_2 in η in the ξ, η-coordinate system, the coordinate transformation must satisfy certain requirements related to both, the convenience of performing the computations and the required accuracy of the numerical solution.

In the examples quoted below we adopt a polar $r\vartheta$-coordinate system in the meridional xy-plane and assume that

$$\xi = 1 - r^{-1}R(\vartheta), \quad \eta = \eta(\vartheta) \tag{1.2}$$

where $R = R(\vartheta)$ is the contour of the streamlined body.

The numerical scheme is based on replacing the system (1.1) by an equivalent one, in which every equation represents the compatibility condition along a special characteristic surface passing through the line $t = \text{const}$, $\xi = \text{const}$. To obtain this scheme we left-multiply (1.1) by the following matrix

$$Q = \left\| \begin{matrix} 1 & \varkappa\rho a \sin\varphi & \varkappa\rho a \cos\varphi & 0 \\ 1 & -\varkappa\rho a \sin\varphi & -\varkappa\rho a \cos\varphi & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \right\|$$

where

$$\sin\varphi = \xi_x(\xi_x^2 + \xi_y^2)^{-1/2}, \quad \cos\varphi = \xi_y(\xi_x^2 + \xi_y^2)^{-1/2} \tag{1.3}$$

denote the spatial components of the normal directions to the characteristic surfaces.

The resulting system of equations is written in the ξ, η-variables as follows

$$Q\frac{\partial Z}{\partial t} + \Lambda Q\frac{\partial Z}{\partial \xi} = -Q(\eta_x A + \eta_y B)\frac{\partial Z}{\partial \eta} - QF \tag{1.4}$$

The matrix Λ appearing here is diagonal and its elements consist of the temporal components of the characteristic normals

$$\Lambda_{11} = u\xi_x + v\xi_y + a(\xi_x^2 + \xi_y^2)^{1/2} \qquad \Lambda_{22} = u\xi_x + v\xi_y - a(\xi_x^2 + \xi_y^2)^{1/2}$$

$$\Lambda_{33} = \Lambda_{44} = u\xi_x + v\xi_y$$

Inequalities $\Lambda_{11} > 0$ and $\Lambda_{22} < 0$ can always be secured by a suitable choice of the functions $\xi(x, y)$. They reflect the fact that the flow along the coordinate ξ is subsonic. The last two elements of the matrix Λ change their sign in the field of flow. The no-leakage condition at the streamlined body, in particular, implies that $\Lambda_{33} = \Lambda_{44} = 0$. We can therefore approximate the left-hand sides of the first two equations of (1.4) using the four-point implicit scheme discussed in [3]; we use the six-point scheme [2] to approximate the derivatives in t and ξ of the remaining equations of (1.4). Generalizing to the second spatial variable is performed analogously to [3].

For the first two equations of the system (1.4) we have therefore

$$\left(\frac{\partial Z}{\partial t}\right)^{n+1/2}_{m+1/2,\, l} = \frac{1}{2\tau}\left(Z^{n+1}_{m+1,l} + Z^{n+1}_{m,l} - Z^{n}_{m+1,l} - Z^{n}_{m,l}\right) -$$

$$- \frac{\sigma_1 k_2}{\tau}\left(Z^{n}_{m+1,l+1} - 2Z^{n}_{m+1,l} + Z^{n}_{m+1,l-1} + Z^{n}_{m,l+1} - 2Z^{n}_{m,l} + Z^{n}_{m,l-1}\right)$$

$$\left(\frac{\partial Z}{\partial \xi}\right)^{n+1/2}_{m+1/2,\, l} = \frac{k_1}{\tau}\left[\alpha\left(Z^{n+1}_{m+1,l} - Z^{n+1}_{m,l}\right) + \beta\left(Z^{n}_{m+1,l} - Z^{n}_{m,l}\right)\right]$$

$$\left(\frac{\partial Z}{\partial \eta}\right)^{n+1/2}_{m+1/2,\, l} = \frac{k_2}{4\tau}\left[\alpha\left(Z^{n+1}_{m+1,l+1} + Z^{n+1}_{m,l+1} - Z^{n+1}_{m+1,l-1} - Z^{n+1}_{m,l-1}\right) +\right.$$

$$\left. + \beta\left(Z^{n}_{m+1,l+1} + Z^{n}_{m,l+1} - Z^{n}_{m+1,l-1} - Z^{n}_{m,l-1}\right)\right]$$

$$Z^{n+1/2}_{m+1/2,l} = \frac{1}{4}\left(Z^{n+1}_{m+1,l} + Z^{n+1}_{m,l} + Z^{n}_{m+1,l} + Z^{n}_{m,l}\right) \qquad (m = 0, 1, \ldots, M-1,\ l = 0, 1, \ldots, L) \tag{1.5}$$

where we assume that

$$\left(\frac{\partial Z}{\partial t}\right)^{n+1/2}_{m,l} = \frac{1}{\tau}\left(Z^{n+1}_{m,l} - Z^{n}_{m,l}\right) - \frac{\sigma_2 k_2}{\tau}\left(Z^{n}_{m,l+1} - 2Z^{n}_{m,l} + Z^{1}_{m,l-1}\right)$$

$$\left(\frac{\partial Z}{\partial \xi}\right)^{n+1/2}_{m,l} = \frac{k_1}{2\tau}\left[\alpha\left(Z^{n+1}_{m+1,l} - Z^{n+1}_{m-1,l}\right) + \beta\left(Z^{n}_{m+1,l} - Z^{n}_{m-1,l}\right)\right]$$

$$\left(\frac{\partial Z}{\partial \eta}\right)^{n+1/2}_{m,l} = \frac{k_2}{2\tau}\left[\alpha\left(Z^{n+1}_{m,l+1} - Z^{n+1}_{m,l-1}\right) + \beta\left(Z^{n}_{m,l+1} - Z^{n}_{m,l-1}\right)\right]$$

$$Z^{n+1/2}_{m,l} = {}^1/_2\left(Z^{n+1}_{m,l} + Z^{n}_{m,l}\right) \qquad (m = 0, 1, \ldots, M;\ l = 0, 1, \ldots, L) \tag{1.6}$$

$$k_i = \tau / h_i, \quad t = n\tau, \quad \alpha + \beta = 1, \quad \alpha > \beta, \quad \sigma_i > 0$$

Here τ is the time interval, while M and L denote the number of intervals in ξ and η. Values of the unknown functions at $l = -1$ and $l = L + 1$ are obtained from the condition of symmetry.

Inserting (1.5) and (1.6) into (1.4) we obtain a system of nonlinear equations. To solve it we shall use the iterative method given in [3]. Following it exactly we take all values from the $(n+1)$-th layer appearing in the coefficients and the right sides of the equations, from the ith iteration. This yields a linear system for computing the required quantities in the $(i+1)$-th iteration on every ray $\eta = \text{const}$.

Using (1.2) to choose the functions $\xi(x, y)$ we find, that $\sin\varphi$ and $\cos\varphi$ are independent of ξ. It follows therefore that the system of equations on the ray splits into three sets which can be written as follows:

$$a_{m+1/2}X_m + b_{m+1/2}X_{m+1} = f_{m+1/2} \qquad (m = 0, 1, \ldots, M-1) \tag{1.7}$$

$$\alpha k_1 c_m T_{m-1} - 2T_m - \alpha k_1 c_m T_{m+1} = g_m \tag{1.8}$$

$$\alpha k_1 c_m S_{m-1} - 2S_m - \alpha k_1 c_m S_{m+1} = h_m \qquad (m = 0, 1, \ldots, M) \tag{1.9}$$

where

$$a = \begin{Vmatrix} \lambda & \varkappa p a \lambda \\ 1 & -\varkappa p a \end{Vmatrix}, \quad b = \begin{Vmatrix} 1 & \varkappa p a \\ \mu & -\varkappa p a \mu \end{Vmatrix}, \quad X = \begin{Vmatrix} p \\ u \sin\varphi + v\cos\varphi \end{Vmatrix}$$

$$\lambda = (1 - 2\alpha k_1 \Lambda_{11}) / (1 + 2\alpha k_1 \Lambda_{11}), \quad \mu = (1 + 2\alpha k_1 \Lambda_{22}) / (1 - 2\alpha k_1 \Lambda_{22})$$

$$c = \Lambda_{33}, \quad T = u\cos\varphi - v\sin\varphi, \quad S = p\rho^{-\varkappa}$$

The boundary conditions for (1.7) are given by

$$(u\sin\varphi + v\cos\varphi)_0 = 0, \quad p_M = 1 \tag{1.10}$$

while those for (1.8) and (1.9) are obtained by equating to zero the coefficient c at $m = 0$ and $m = M$.

By [4] the boundary value problem for systems (1.8) and (1.9) is well defined. It can be solved using the double sweep method investigated in detail in [5].

The boundary value problem (1.7).(1.10) is well defined [4] if $|\lambda| < 1$ and $|\mu| < 1$. The latter conditions hold if and only if $\Lambda_{11} > 0$ and $\Lambda_{22} < 0$, and the above inequalities become invalid only when $m = M$, in which case $\Lambda_{11} = \Lambda_{22} = 0$. We may therefore expect some loss of accuracy in computations when the values of ξ differ little from unity. Physically this is associated with the problem of mapping an infinite region into a finite one, and the fact that large distances from the body affect the approximation adversely. Moreover, at $\xi = 1$ the velocity of the flow must be constant and equal to M_∞, and this makes the problem overdefined. A separate computation using the boundary condition $(u\sin\varphi + v\cos\varphi)_M = M_\infty \sin\varphi$ in place of the second equation of (1.10) has shown, that the results begin to diverge not earlier than in the fourth place. Problems (1.7) and (1.10) were solved using the double sweep method given in [3].

2. The above method was used to compute transonic flows past various axisymmetric bodies.

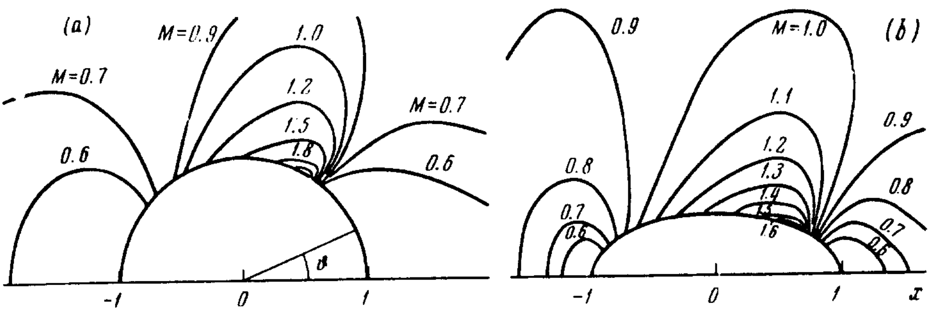

Fig. 1

Figure 1a depicts Mach number level curves for a sphere in a flow whose $M_\infty = 0.8$. We see that a region of accumulation of the curves $M = \text{const}$ appears behind the sphere and we assume that a shock wave exists in this region. Figure 2 depicts the pressure distribution on the sphere for various values of M_∞. Here we see that the pressure varies almost discontinuously. It is interesting to note that the overall pressure loss in normal shock when the computed value of M is assumed in front of this shock, is nearly equal to the change of stagnation pressure at the front and rear points of the sphere. A check that the Bernoulli's integral is valid gives the maximum error in the vicinity of the shock. Although this error increases with the strength of the shock, it does not exceed 6 % when $M_\infty = 0.9$. Crosses appearing on the graphs denote the sonic points.

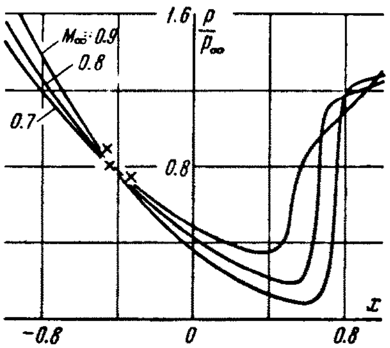

Fig. 2

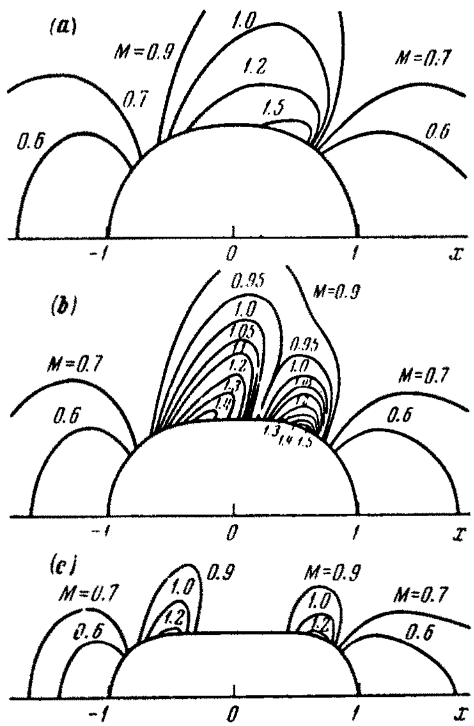

Fig. 3

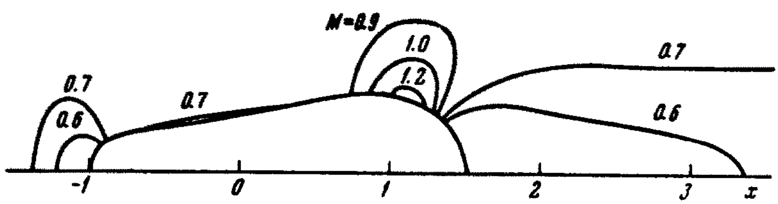

Fig. 4

The results obtained indicate that the flows with weak shocks can be computed directly using nondivergent implicit schemes.

Figure 1b shows the pattern of lines $M = \text{const}$ formed when an ellipsoid of revolution with the aspect ratio $= 0.5$ is streamlined by a flow of $M_\infty = 0.95$. The general flow pattern resembles that shown on Fig. 1a. The shock strength diminishes with increasing distance from the body more rapidly, than in the case of a flow past a sphere.

Figure 3 gives the fields of flow past a combination of two spheres and a cylinder, at $M_\infty = 0.8$, for various values of the aspect ratio. Evolution of the flow is well illustrated. On Fig. 3a the body is nearly spherical and a single supersonic zone appears in its vicinity. On increasing the length of the cylinder (Fig. 3b) the supersonic zone splits into two distinct zones, the second of which is situated downstream and contains a stronger shock than the first one, although the shock is still weaker than that appearing in Fig. 3a. Figure 3c depicts the case when the cylindrical part of the body is still longer. Here two weak supersonic zones appear which are spaced even further apart.

Figure 4 shows a flow past a combination of two spheres and a 10% cone, again at $M_\infty = 0.8$. Here the supersonic zone is situated at the rear part of the body. The flow first accelerates on the front sphere reaching $M \approx 0.8$, then slows down to $M \approx 0.66$ and flows past the cone with very slowly increasing velocity.

Computations are also performed for a flow past a 10% spherically truncated cone with various ellipsoidal tailpieces. The distribution of parameters along the body up to some small distance from the point of attachment of the ellipsoid are practically identical with those of the case shown on Fig. 4.

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NONSYMMETRIC MECHANICS OF TURBULENT FLOWS

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In [1] we proposed renouncing the hypothesis of a symmetric tensor of Reynolds stresses and an agitated fluid and introducing an equation of conservation of the moment of momentum. This equation turns out to be nontrivial if, for example, the pulsed momentum transfer through a flow cross section depends on the orientation of the cross section in space.

In the present paper we derive the equations of nonsymmetrical mechanics of turbu-